FROBENIUS OBJECTS IN CARTESIAN BICATEGORIES

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Abstract. Maps (left adjoint arrows) between Frobenius objects in a cartesian bicategory $\mathbf{B}$ are precisely comonoid homomorphisms and, for A Frobenius and any T in $\mathbf{B}$, $\mathrm{Map}(\mathbf{B})(T, A)$ is a groupoid.

1. Introduction

The notion of locally ordered cartesian bicategory was introduced by Carboni and Walters [C&W] for the axiomatization of the bicategory of relations of a regular category. The notion has since been extended by Carboni, Kelly, Walters, and Wood [CKWW] to the case of a general bicategory, to include examples such as bicategories of spans, cospans, and profunctors.

A crucial further axiom introduced by Carboni and Walters in that paper was the so-called discreteness axiom, now known as the Frobenius axiom, since it was recognized to be equivalent to Lawvere's equational version [LAW] of Frobenius algebra. With this axiom one can define the notion of Frobenius object in a monoidal category, the Frobenius axiom being an equation satisfied by monoid and comonoid structures on the object.

The Frobenius axiom has found a large variety of uses. For example, the 2-dimensional cobordism category has been shown to be the symmetric monoidal category with a generic commutative Frobenius object. (For a presentation of this result see J. Kock [Ko].) Related results are the characterization of the symmetric monoidal category of cospans of finite sets in [LACK] and the characterization of the symmetric monoidal category of cospans of finite graphs in [RSW]. Another example is that, in the algebra of quantum measurement [Co&P], classical data types are Frobenius objects. In [G&H] the Frobenius equation is a crucial equation in an algebraic presentation of double pushout graph rewriting, and in [KaSW] the equation is one of the main equations in a compositional theory of automata. The 2-dimensional version of Frobenius algebra has also been introduced in the characterization of a certain monoidal 2-category in [MSW].

There is a rather obvious way of extending the notion of Frobenius object to the context of a monoidal bicategory: instead of requiring equations between operations, certain canonical 2-cells are required to be invertible. This paper develops properties of such 2-dimensional Frobenius objects, for the canonical monoid and comonoid structure on each object which is part of the cartesian bicategory structure. The two principal

The authors gratefully acknowledge financial support from the Italian CNR and the Canadian NSERC. Diagrams typeset using M. Barr's diagram package, diagxy.tex.

2000 Mathematics Subject Classification: 18A25.

Key words and phrases: cartesian bicategory, Frobenius object, dual object, groupoid.

results are (i) that maps (left adjoint arrows) between Frobenius objects are the same as comonoid homomorphisms, and (ii) that if A is a Frobenius object then, for any object T in the cartesian bicategory $\mathbf{B}$, $\mathrm{Map}(\mathbf{B})(T,A)$ is a groupoid. This second result was noticed for the special case of Profunctors at the time of the Carboni-Walters paper by Carboni and Wood, independently, but has never been published. We develop in this paper techniques in a general cartesian bicategory which enable us to lift the profunctor proof.

The results of this paper will be used in a following paper [W&W] characterizing bicategories of spans.

We thank Bob Paré for his helpful comments when these results were presented at the ATCAT seminar in Halifax, Canada.

2. Preliminaries

2.1. We recall from [CKWW] that a bicategory $\mathbf{B}$ is *cartesian* if the subbicategory of maps (by which we mean left adjoint arrows), $\mathbf{M} = \mathrm{Map}\mathbf{B}$, has finite products $(- \times -, 1)$ with projections denoted $p : X \leftarrow X \times Y \rightarrow Y : r$; each hom-category $\mathbf{B}(X,A)$ has finite products $(- \wedge -, \top)$ with projections denoted $\pi : R \leftarrow R \wedge S \rightarrow S : \rho$; and an evident derived tensor product on $\mathbf{B}$, $(- \otimes -, I)$ extending the product structure of $\mathbf{M}$, is functorial. It was shown that the derived tensor product of a cartesian bicategory underlies a symmetric monoidal bicategory structure. Throughout this paper, $\mathbf{B}$ is assumed to be a cartesian bicategory and, as in [CKWW], we assume, for ease of notation, that $\mathbf{B}$ is normal, meaning that the identity compositional constraints of $\mathbf{B}$ are identity 2-cells.

2.2. If f is a map of $\mathbf{B}$, an arrow of $\mathbf{M}$, we will write $\eta_f, \epsilon_f : f \dashv f^*$ for a chosen adjunction in $\mathbf{B}$ that makes it so. We occasionally refer to an f^* as a *pam*. As in [CKWW], we write

$$\begin{tikzcd} & \mathbf{G} \arrow[dl, "\partial_0"'] \arrow[dr, "\partial_1"] & \\ \mathbf{M} & & \mathbf{M} \end{tikzcd}$$

for the Grothendieck span corresponding to

$$\mathbf{M}^{\mathrm{op}} \times \mathbf{M} \xrightarrow{i^{\mathrm{op}} \times i} \mathbf{B}^{\mathrm{op}} \times \mathbf{B} \xrightarrow{\mathbf{B}(-,-)} \mathbf{CAT}$$

where $i : \mathbf{M} \rightarrow \mathbf{B}$ is the inclusion. A typical arrow of $\mathbf{G}$, $(f, \alpha, u) : (X, R, A) \rightarrow (Y, S, B)$ can be depicted by a square in $\mathbf{B}$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "R"'] \arrow[dr, phantom, "\xrightarrow{\alpha}"] & Y \arrow[d, "S"] \\ A \arrow[r, "u"'] & B \end{tikzcd} \tag{1}$$

in which f and u are maps, and such arrows are composed by pasting. A 2-cell (ϕ,ψ): $(f,\alpha,u){\to}(g,\beta,v)$ in $\mathbf{G}$ is a pair of 2-cells $\phi{:}f{\to}g$, $\psi{:}u{\to}v$ in $\mathbf{M}$ which satisfy the obvious equation.

2.3. In part of this and subsequent work it will be useful to revisit certain of the arrows of $\mathbf{G}$ from another point of view. Consider

$$\begin{tikzcd} T \arrow[r, "1_T"] \arrow[d, "x"'] \arrow[dr, phantom, "\Downarrow\rho"] & T \arrow[d, "y"] \\ X \arrow[r, "R"'] & Y \end{tikzcd}$$

On the one hand it is just an arrow from 1_T to R in $\mathbf{G}$ but each of the three reformulations of ρ that result from taking mates have their uses.

$$\begin{tikzcd} T \arrow[r, "1_T"] \arrow[d, "x"'] \arrow[dr, phantom, "\Downarrow\widehat{\rho}"] & T \\ X \arrow[r, "R"'] & Y \arrow[u, "y^*"'] \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "1_T"] \arrow[dr, phantom, "\Downarrow\rho^*"] & T \\ X \arrow[u, "x^*"] \arrow[r, "R"'] & Y \arrow[u, "y^*"'] \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "1_T"] \arrow[dr, phantom, "\Downarrow\widetilde{\rho}"] & T \arrow[d, "y"] \\ X \arrow[u, "x^*"] \arrow[r, "R"'] & Y \end{tikzcd}$$

In the first of these, $\widehat{\rho}{:}\,1_T{\to}y^*Rx$, it is sometimes convenient to write $R(y,x) = y^*Rx$ and regard $\widehat{\rho}$ as a 1_T-element of $R(y,x)$. In the special case where R is $1_X: X{\to}X$ we write $X(y,x) = y^*x$ (invoking normality of $\mathbf{B}$). (This hom-notation is similar to that employed first in [S&W]. It was adapted for this compositional context in [Wd].) The second we will use without further comment except to say that, for $R = 1_X$, ρ^* is the usual way of making the process of taking right ajoints functorial. The third will appear in our discussion of tabulations in the forthcoming [W&W]. Note that the $R(y,x)$ notation extends to 2-cells so that, for $\eta{:}y'{\to}y$ and $\xi{:}x{\to}x'$, we have $R(\eta,\xi){:}R(y,x){\to}R(y',x')$.

The chief purpose of the notation $R(y,x)$ is to guide intuition so that constructions in such cartesian bicategories as that of categories, profunctors, and equivariant 2-cells (which we call **prof**) can be usefully generalized. Observe that if $\tau{:}R{\to}S$ is a 2-cell in $\mathbf{B}$ and $\xi{:}x{\to}x'$ then we have automatically such identities as $\tau(y,x').R(y,\xi) = S(y,\xi).\tau(y,x)$, both providing the horizontal composite $\tau\xi$ whiskered with y^* as below.

$$\begin{tikzcd} T \arrow[r, bend left=50, "x"] \arrow[r, bend right=50, "x'"'] \arrow[r, phantom, "\Downarrow\xi"] & X \arrow[r, bend left=50, "R"] \arrow[r, bend right=50, "S"'] \arrow[r, phantom, "\Downarrow\tau"] & Y \arrow[r, "y^*"] & T \end{tikzcd}$$

For the most part, we will use such calculations with little comment.

If

$$\begin{tikzcd} T \arrow[r, "1_T"] \arrow[d, "x"'] \arrow[dr, phantom, "\Downarrow\widehat{\rho}"] & T \\ X \arrow[r, "R"'] & Y \arrow[u, "y^*"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} T \arrow[r, "1_T"] \arrow[d, "y"'] \arrow[dr, phantom, "\Downarrow\widehat{\sigma}"] & T \\ Y \arrow[r, "S"'] & Z \arrow[u, "z^*"'] \end{tikzcd}$$

are 1_T-elements of $R(y,x)$ and $S(z,y)$ respectively then it is easy to see that $\widehat{\rho\square\sigma}$, where $\rho\square\sigma$ is the paste composite of ρ and σ, is a 1_T-element of $(SR)(z,x)$. The 1_T-element $\widehat{\rho\square\sigma}$ can be given in several ways. We will have occasion to give it via the pasting composite

$$\begin{tikzcd} T \arrow[rr, "1_T"] \arrow[d, "x"'] \arrow[drr, phantom, "\rho\Downarrow"] & & T \arrow[rr, "1_T"] \arrow[dr, "y"'] \arrow[rr, phantom, shift right=5, "\Downarrow\eta_y"] & & T \arrow[rr, "1_T"] \arrow[drr, phantom, "\Downarrow\sigma^*"] & & T \\ X \arrow[rrr, "R"'] & & & Y \arrow[ur, "y^*"'] \arrow[rrr, "S"'] & & & Z \arrow[u, "z^*"'] \end{tikzcd}$$

We note that a paste composite such as $\rho\square\sigma$ as below

$$\begin{tikzcd} T \arrow[r, "1_T"] \arrow[d, "x"'] \arrow[dr, phantom, "\rho\Downarrow"] & T \arrow[r, "1_T"] \arrow[d, "y"] \arrow[dr, phantom, "\Downarrow\sigma"] & T \arrow[d, "z"] \\ X \arrow[r, "R"'] & Y \arrow[r, "S"'] & Z \end{tikzcd}$$

may result from several different $y{:}T \to Y$. For example, in

$$\begin{tikzcd} T \arrow[r, "1_T"] \arrow[d, "x"'] \arrow[dr, phantom, "\rho\Downarrow"] & T \arrow[r, "1_T"] \arrow[d, bend right=50, "y"'] \arrow[d, bend left=50, "y'"] \arrow[d, phantom, "\overset{\eta}{\Leftarrow}"] \arrow[dr, phantom, "\Downarrow\sigma"] & T \arrow[d, "z"] \\ X \arrow[r, "R"'] & Y \arrow[r, "S"'] & Z \end{tikzcd}$$

we have $(\rho\square\eta)\square\sigma = \rho\square(\eta\square\sigma)$ suggesting that some of the 1_T-elements of $(SR)(z,x)$ are given by an obvious coend over y in the category $\mathbf{M}(T,Y)$.

However, our **prof**-like notation has its limitations. For fixed T we can associate to X the category $\widetilde{X} = \mathbf{M}(T,X)$ and to $R : X \to Y$ the profunctor $\widetilde{R} : \widetilde{X} \to \widetilde{Y}$ where $\widetilde{R}(y,x) = \mathbf{B}(T,T)(1_T, y^*Rx)$ but we see no reason why a general 1_T-element of $(SR)(z,x)$ in a general cartesian bicategory should arise from pasting a 1_T-element of $S(z,y)$ to a 1_T-element of $R(y,x)$ for some $y : T \to Y$. In short, while there is a 2-cell $\widetilde{S}\widetilde{R} \to \widetilde{SR}$ in **prof** there seems to be no reason why it should have surjective components. That said, $\widetilde{S}\widetilde{R} \to \widetilde{SR}$ *is* an isomorphism in case $\mathbf{B} = \mathrm{Span}\mathcal{E}$, for any category $\mathcal{E}$ with finite limits, and

for any cartesian $\mathbf{B}$ we have isomorphisms $1_{\widetilde{X}} \to \widetilde{1_X}$ in **prof**, for any X in $\mathbf{B}$. So there is always a normal lax functor

$$\widetilde{(-)}:\mathbf{B} \to \mathbf{prof}$$

which in *some* cases is a pseudofunctor. Fortunately, we have no need for invertibility of the $\widetilde{S}\widetilde{R} \to \widetilde{SR}$.

2.4. Quite generally, an arrow of $\mathbf{G}$ as given by the square (1) will be called a *commutative* square if α is invertible. The arrow (1) of $\mathbf{G}$ will be said to satisfy the *Beck-Chevalley condition* if the mate of α under the adjunctions $f \dashv f^*$ and $u \dashv u^*$, as given in the square below (no longer an arrow of $\mathbf{G}$), is invertible.

$$\begin{tikzcd}
X \arrow[d, "R"'] \arrow[r, phantom, ""] & Y \arrow[l, "f^*"'] \arrow[d, "S"] \\
A \arrow[ur, phantom, "\xrightarrow{\alpha^*}"] & B \arrow[l, "u^*"]
\end{tikzcd}$$

Thus Proposition 4.8 of [CKWW] says that projection squares of the form $\tilde{p}_{R,1_Y}$ and $\tilde{r}_{1_X,S}$ satisfy the Beck-Chevalley condition. (Also, Proposition 4.7 of [CKWW] says that the same projection squares are commutative. In general, neither commutative nor Beck-Chevalley implies the other.) If R and S are also maps and α is invertible then α^{-1} gives rise to another arrow of $\mathbf{G}$ which may or may not satisfy the Beck-Chevalley condition. The point here is that a commutative square of maps gives rise to two, generally distinct, Beck-Chevalley conditions. It is well known that, for bicategories of the form Span$\mathcal{E}$ and Rel$\mathcal{E}$ all pullback squares of maps satisfy both Beck-Chevalley conditions. A [bi]category with finite products has automatically a number of pullbacks which we might call *product-absolute* pullbacks because they are preserved by all [pseudo]functors which preserve products.

3. Frobenius Objects in Cartesian Bicategories

For any object A in $\mathbf{B}$, we have the following two $\mathbf{G}$ arrows:

$$\begin{tikzcd}
A \arrow[r, "d"] \arrow[dd, "d"'] & A \otimes A \arrow[d, "d\otimes 1"] \\
\arrow[r, phantom, "\xrightarrow{=}"] & (A \otimes A) \otimes A \arrow[d, "a"] \\
A \otimes A \arrow[r, "1\otimes d"'] & A \otimes (A \otimes A)
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rr, "d"] \arrow[d, "d"'] \arrow[drr, phantom, "\xrightarrow{=}"] & & A \otimes A \arrow[d, "1\otimes d"] \\
A \otimes A \arrow[r, "d\otimes 1"'] & (A \otimes A) \otimes A \arrow[r, "a"'] & A \otimes (A \otimes A)
\end{tikzcd}$$

obtained from the same equality of arrows in Map$\mathbf{B}$. (With a suitable choice of conventions we have equality rather than a mere isomorphism.) For each square, observe that the data regarded as a square in $\mathbf{M}$ provide an example of a product-absolute pullback.

3.1. Definition. *An object A is said to be* Frobenius *if both of the* $\mathbf{G}$ *arrows above satisfy the Beck-Chevalley condition. This is to demand invertibility both of $\delta_0 : d.d^* \to 1_A \otimes d^*.a.d \otimes 1_A$, the mate of the first equality above, and of $\delta_1 : d.d^* \to d^* \otimes 1_A.a^*.1_A \otimes d$, the mate of the second equality above.*

3.2. Lemma. *The Beck-Chevalley condition for either square implies the condition for the other.*

Proof. Explicitly, in notation suppressing $\otimes$, δ_0 and δ_1 are given by

$$\delta_0 \;=\; \begin{tikzcd} AA \arrow[rr, "1"] \arrow[dr, "d^*"'] & & AA \arrow[r, "dA"] & (AA)A \arrow[r, "a"] & A(AA) \arrow[dr, "Ad^*"] \arrow[d, phantom, "\Uparrow A\eta"] & \\ & A \arrow[u, phantom, "\Uparrow\epsilon"] \arrow[ur, "d"'] \arrow[rr, "d"'] & & AA \arrow[u, phantom, "\Uparrow"] \arrow[ur, "Ad"] \arrow[rr, "1"'] & & AA \end{tikzcd}$$

and

$$\delta_1 \;=\; \begin{tikzcd} AA \arrow[rr, "1"] \arrow[dr, "d^*"'] & & AA \arrow[r, "Ad"] & A(AA) \arrow[r, "a^*"] & (AA)A \arrow[dr, "d^*A"] \arrow[d, phantom, "\Uparrow \eta A"] & \\ & A \arrow[u, phantom, "\Uparrow\epsilon"] \arrow[ur, "d"'] \arrow[rr, "d"'] & & AA \arrow[u, phantom, "\Uparrow"] \arrow[ur, "dA"] \arrow[rr, "1"'] & & AA \end{tikzcd}$$

Assume that δ_0 is invertible and paste at its top and right edges the following pasting composite at its bottom edge.

$$\begin{tikzcd} & & & A(AA) \arrow[r, "a^*"] & (AA)A \arrow[dr, "sA"] & & \\ AA \arrow[r, "1"] & AA \arrow[r, "Ad"] & A(AA) \arrow[ur, "As"] & & & (AA)A \arrow[r, "d^*A"] & AA \\ AA \arrow[u, "s"'] \arrow[r, "1"'] \arrow[ur, phantom, "\cong"] & AA \arrow[u, "s"'] \arrow[r, "dA"'] \arrow[ur, phantom, "\cong"] & (AA)A \arrow[u, "s"'] \arrow[rrr, "a"'] \arrow[urrr, phantom, "\cong"] & & & A(AA) \arrow[u, "s"'] \arrow[r, "Ad^*"'] \arrow[ur, phantom, "\cong"] & AA \arrow[u, "s"'] \end{tikzcd}$$

The squares are pseudonaturality squares for symmetry as in 4.5 of [CKWW] and the hexagon bounds an invertible modification constructed from those relating the associativity equivalence a and the symmetry equivalence s. Next, observe that we have $sd \cong d$ and, since s is an equivalence with $s^*_{A,B} \cong s_{B,A}$, $d^*s \cong d^*$. By functoriality of $\otimes$ we have also $(As)(Ad) \cong Ad$ and $(d^*A)(sA) \cong d^*A$. Noting the compatibility of the pseudonatural transformation s with the 2-cell ηA, the large pasting composite is seen to be δ_1. The derivation of invertibility of δ_0 from that of δ_1 is effected in a similar way. ∎

3.3. Axiom. Frobenius *A cartesian bicategory* $\mathbf{B}$ *is said to satisfy the* Frobenius *axiom if, for each A in $\mathbf{B}$, A is Frobenius.*

3.4. Proposition. *In a cartesian bicategory $\mathbf{B}$, the Frobenius objects are closed under finite products.*

Proof. Consider a Frobenius object A so that we have invertible $\delta_0 = \delta_0(A)$ in

$$\begin{tikzcd}
A \arrow[d, "d"'] \arrow[rr, phantom, ""] & & A \otimes A \arrow[ll, "d^*"'] \arrow[d, "d\otimes 1"] \\
{} \arrow[r, phantom, "\xrightarrow{\delta_0}"] & & (A \otimes A) \otimes A \arrow[d, "a"] \\
A \otimes A & & A \otimes (A \otimes A) \arrow[ll, "1\otimes d^*"]
\end{tikzcd}$$

For B also Frobenius, form the tensor product of the diagrams for $\delta_0(A)$ and $\delta_0(B)$, noting that $\delta_0(A)\otimes\delta_0(B)$ is also invertible. The diagram for $\delta_0(A\otimes B)$ is easily formed from that of $\delta_0(A)\otimes\delta_0(B)$ by pasting to its exterior the requisite permutations of the A and B and using such isomorphisms as $m(d_A \otimes d_B) \cong d_{A\otimes B}$, where $m{:}(A\otimes A)\otimes(B\otimes B)\to(A\otimes B)\otimes(A\otimes B)$ is the middle-four interchange equivalence. Thus $A\otimes B$ is Frobenius when A and B are so. Invertibility of $\delta_0(I)$ follows easily since d_I is an equivalence, showing that I is Frobenius. ∎

Write Frob$\mathbf{B}$ for the full subbicategory of $\mathbf{B}$ determined by the Frobenius objects. It follows immediately from Proposition 3.4 that

3.5. Proposition. *For a cartesian bicategory $\mathbf{B}$, the full subbicategory* Frob$\mathbf{B}$ *is a cartesian bicategory which satisfies the Frobenius axiom.* ∎

In any (pre)cartesian bicategory we have, for each object X, the following arrows:

$$N_X \quad = \quad I \xrightarrow{t_X^*} X \xrightarrow{d_X} X \otimes X \qquad \text{and} \qquad E_X \quad = \quad X \otimes X \xrightarrow{d_X^*} X \xrightarrow{t_X} I$$

Since the cartesian bicategory $\mathbf{B}$ is a (symmetric) monoidal bicategory it can be seen as a one-object tricategory, so that pseudo adjunctions $N, E\colon X \dashv A$, where X and A are *objects* of $\mathbf{B}$ (and N and E are arrows of $\mathbf{B}$), are well defined. (We note that, especially since $\mathbf{B}$ is symmetric, it is customary to speak of such X and A as *duals*.)

3.6. Proposition. *For a Frobenius object X in a cartesian bicategory, N_X and E_X provide the unit and counit for a pseudo-adjunction $X \dashv X$.*

Proof. (Sketch) We are to exhibit isomorphisms

$$(E_X \otimes X)a^*(X \otimes N_X) \cong s_{X,I} \quad \text{and} \quad (X \otimes E_X)a(N_X \otimes X) \cong s_{I,X}$$

subject to two coherence equations. Consider:

$$
\begin{tikzcd}
X \otimes I \arrow[r, "X\otimes t_X^*"] \arrow[dr, "r"'] \arrow[r, phantom, shift right=4ex, "\cong"] & X \otimes X \arrow[r, "X\otimes d_X"] \arrow[d, "d_X^*"] & X \otimes (X \otimes X) \arrow[r, "a^*"] \arrow[d, phantom, "\overset{\delta_1}{\underset{\cong}{\longrightarrow}}"] & (X \otimes X) \otimes X \arrow[d, "d_X^* \otimes X"] \\
 & X \arrow[rr, "d_X"'] \arrow[drr, "l"'] & \arrow[dr, phantom, "\cong"] & X \otimes X \arrow[d, "t_X \otimes X"] \\
 & & & I \otimes X
\end{tikzcd}
$$

$$
\begin{tikzcd}
I \otimes X \arrow[r, "t_X^* \otimes X"] \arrow[dr, "l"'] \arrow[r, phantom, shift right=4ex, "\cong"] & X \otimes X \arrow[r, "d_X \otimes X"] \arrow[d, "d_X^*"] & (X \otimes X) \otimes X \arrow[r, "a"] \arrow[d, phantom, "\overset{{\delta_0}^{-1}}{\underset{\cong}{\longleftarrow}}"] & X \otimes (X \otimes X) \arrow[d, "X \otimes d_X^*"] \\
 & X \arrow[rr, "d_X"'] \arrow[drr, "r"'] & \arrow[dr, phantom, "\cong"] & X \otimes X \arrow[d, "X \otimes t_X"] \\
 & & & X \otimes I
\end{tikzcd}
$$

For the coherence requirements let us abbreviate $\otimes$ by juxtaposition, as we have before, but now work as if the bicategory constraints of $\mathbf{B}$ and those of the monoidal structure $(\mathbf{B}, \otimes, I)$ are strict. (In general, this is not acceptable because a monoidal bicategory is not tri-equivalent to a one-object 3-category. However, our monoidal structure, being given by universal properties, is less problematical.) Temporarily, write $N : I \to X^\circ X$ and $E : XX^\circ \to I$, just to mark the role of the X's. Write $\alpha : 1_X \to (EX)(XN)$ and $\beta : (X^\circ E)(NX^\circ) \to 1_{X^\circ}$ for the isomorphisms built from those above, with the simplifying assumptions. The coherence requirements of α and β are the pasting equations

$$
1_E = \quad
\begin{tikzcd}
XX^\circ \arrow[dr, "XNX^\circ"] \arrow[drr, bend left=40, "XX^\circ"] \arrow[dd, bend right=40, "XX^\circ"'] \arrow[drr, phantom, shift left=2ex, "\xrightarrow{X\beta}"] & & \\
\arrow[r, phantom, "\xrightarrow{\alpha X^\circ}"] & XX^\circ XX^\circ \arrow[r, "XX^\circ E"] \arrow[d, "EXX^\circ"'] \arrow[dr, phantom, "\cong"] & XX^\circ \arrow[d, "E"] \\
 & XX^\circ \arrow[r, "E"'] & I
\end{tikzcd}
\qquad
1_N = \quad
\begin{tikzcd}
I \arrow[r, "N"] \arrow[d, "N"'] \arrow[dr, phantom, "\cong"] & X^\circ X \arrow[d, "NX^\circ X"] \arrow[ddr, bend left=40, "X^\circ X"] & \\
X^\circ X \arrow[r, "X^\circ XN"'] \arrow[drr, bend right=40, "X^\circ X"'] \arrow[dr, phantom, "\xrightarrow{X^\circ\alpha}"] & X^\circ X X^\circ X \arrow[dr, "X^\circ EX"] \arrow[r, phantom, "\xrightarrow{\beta X}"] & {} \\
 & & X^\circ X
\end{tikzcd}
$$

where the unlabelled isomorphisms in the squares are given by pseudofunctoriality of $\otimes$. We will verify the first of these equations, verification of the second being similar, now using $X^\circ = X$ but continuing to supress the constraints both for $\mathbf{B}$ and for the monoidal structure. Thus we must show that the composite on the left below

$$
\begin{tikzcd}
XX \arrow[dr, "Xt^*X"] & & & & \\
& XXX \arrow[r, "Xd^*"] \arrow[dd, "d^*X"'] \arrow[dr, "XdX"] & XX \arrow[dr, "Xd"] \arrow[d, phantom, "\xrightarrow{X\delta_0^{-1}}"] & & \\
& & XXXX \arrow[r, "XXd^*"] \arrow[dd, "d^*XX"] & XXX \arrow[dd, "d^*X"] & \\
& XX \arrow[r, phantom, "\xrightarrow{\delta_1 X}"] \arrow[dr, "dX"'] & {} \arrow[r, phantom, "\cong"] & {} & \\
& & XXX \arrow[r, "Xd^*"'] & XX \arrow[dr, "tt"] & \\
& & & & I
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
XX \arrow[dr, "Xt^*X"] & & & & & \\
& XXX \arrow[rr, "Xd^*"] \arrow[dd, "d^*X"'] \arrow[ddrr, phantom, "\cong"] & & XX \arrow[dd, "d^*"] \arrow[dr, "Xd"] & & \\
& & & & XXX \arrow[dd, "d^*X"] & \\
& XX \arrow[rr, "d^*"] \arrow[dr, "dX"'] \arrow[drrr, phantom, "\xrightarrow{\delta_0^{-1}}"] & & X \arrow[r, phantom, "\xrightarrow{\delta_1}"] \arrow[dr, "d"] & {} & \\
& & XXX \arrow[rr, "Xd^*"'] & & XX \arrow[dr, "tt"] & \\
& & & & & I
\end{tikzcd}
$$

is 1_E. Again using pseudofunctoriality of $\otimes$, we have the equality shown and finally the diagram on the right can be shown to be 1_E from the definitions of δ_0 and δ_1. ■

3.7. If $R{:}X{\to}A$ is an arrow in $\mathbf{B}$ then given pseudo adjunctions $X \dashv X^\circ$ and $A \dashv A^\circ$ we should expect that adaption of the calculus of *mates* found in [K&S] will enable us to define $R^\circ{:}X^\circ{\to}A^\circ$ by the usual formula. In fact, if every object of $\mathbf{B}$ has a dual one should expect $(-)^\circ$ to provide a pseudofunctor $(-)^\circ : \mathbf{B}^{\mathrm{oprev}}{\to}\mathbf{B}$ between tricategories, where $(-)^{\mathrm{rev}}$ denotes dualization with respect to objects of $\mathbf{B}$ *composed* via $\otimes$, while as usual $(-)^{\mathrm{op}}$ denotes dualization with respect to the 1-cells of $\mathbf{B}$. In particular, one should expect $(X \otimes Y)^\circ \simeq Y^\circ \otimes X^\circ$. The point of this paragraph is that the $(-)^\circ$ of the following proposition arises from the properties already under consideration and is not a new structure as in the similarly denoted operation of [F&S].

3.8. Proposition. *For a cartesian bicategory* $\mathbf{B}$ *in which every object is Frobenius, there is an involutory pseudofunctor*

$$(-)^\circ{:}\mathbf{B}^{\mathrm{op}}{\to}\mathbf{B}$$

which is the identity on objects.

Proof. With $X^\circ = X$ we define

$$(-)^\circ_{A,X}{:}\mathbf{B}^{\mathrm{op}}(A,X) = \mathbf{B}(X,A){\to}\mathbf{B}(A,X)$$

by the evidently functorial formula

$$R^\circ = (X \otimes E_A)(X \otimes R \otimes A)(N_X \otimes A)$$

In terms of the one object tricategory $(\mathbf{B}, \otimes, I)$ with single object $*$, we can express R° by the pasting

$$\begin{tikzcd}
\ast \arrow[rr] \arrow[dr, "A"'] & & \ast \arrow[rr, "I"] & & \ast \arrow[dr, "X"] \arrow[d, phantom, "\Uparrow N_X"] & \\
& \ast \arrow[ur, "A"'] \arrow[rr, "I"'] \arrow[u, phantom, "\Uparrow E_A"] & & \ast \arrow[ur, "X"] \arrow[rr, "I"'] \arrow[u, phantom, "\Uparrow R"] & & \ast
\end{tikzcd}$$

For $R{:}X{\to}A$, along with $S{:}A{\to}Y$, to give $\widetilde{(-)^\circ}{:}R^\circ S^\circ{\to}(SR)^\circ$ we consider

$$\begin{tikzcd}
\ast \arrow[rr, "I"] \arrow[dr, "Y"'] & & \ast \arrow[rr, "I"] & & \ast \arrow[rr, "I"] \arrow[dr, "A"] \arrow[d, phantom, "\Uparrow N_A"] & & \ast \arrow[rr, "I"] & & \ast \arrow[dr, "X"] \arrow[d, phantom, "\Uparrow N_X"] & \\
& \ast \arrow[ur, "Y"'] \arrow[rr, "I"'] \arrow[u, phantom, "\Uparrow E_Y"] & & \ast \arrow[ur, "A"] \arrow[rr, "I"'] \arrow[u, phantom, "\Uparrow S"] & & \ast \arrow[ur, "A"'] \arrow[rr, "I"'] \arrow[u, phantom, "\Uparrow E_A"] & & \ast \arrow[ur, "X"] \arrow[rr, "I"'] \arrow[u, phantom, "\Uparrow R"] & & \ast
\end{tikzcd}$$

in which the pasting composite displays $R^\circ S^\circ$. The required $\widetilde{(-)^\circ}$ is obtained as the collapsing of the centre triangles using $\alpha^{-1} : (E_A \otimes A)(A \otimes N_A) \cong s_{A,I}$ of the pseudo adjunction $N_A, E_A{:}A \dashv A$. Evidently, $\widetilde{(-)^\circ}$ is invertible. We give the identity constraint for $(-)^\circ$ as $\beta^{-1}{:}1_X{\to}(X{\otimes}E_X)(N_X{\otimes}X)$ which is again invertible. Finally, having observed that the mate description of $R^\circ = (X \otimes E_A)(X \otimes R \otimes A)(N_X \otimes A)$ was given by expanding $R{:}X{\to}A$ as $R{:}X \otimes I{\to}I \otimes A$ we see by writing $R{:}I \otimes X{\to}A \otimes I$ that we have equally

$$R^\circ \cong \ (E_A \otimes X)(A \otimes R \otimes X)(A \otimes N_X)$$

Thus we may as well give

$$((-)^\circ)^{\mathrm{op}}{:}\mathbf{B}{\to}\mathbf{B}^{\mathrm{op}}$$

by the formula

$$(A \xrightarrow{S} X){\mapsto}(E_X \otimes A)(X \otimes S \otimes A)(X \otimes N_A)$$

so that $R^{\circ\circ}$ is the pasting

$$\begin{tikzcd}
& \ast \arrow[rr, "I"] \arrow[dr, "A"] \arrow[d, phantom, "\Uparrow N_A"] & & \ast \arrow[rr, "I"] & & \ast \arrow[rr, "I"] \arrow[dr, "X"'] \arrow[d, phantom, "\Uparrow N_X"] & & \ast \\
\ast \arrow[ur, "A"] \arrow[rr, "I"'] & & \ast \arrow[ur, "A"'] \arrow[rr, "I"'] \arrow[u, phantom, "\Uparrow E_A"] & & \ast \arrow[ur, "X"] \arrow[rr, "I"'] \arrow[u, phantom, "\Uparrow R"] & & \ast \arrow[ur, "X"'] \arrow[u, phantom, "\Uparrow E_X"] &
\end{tikzcd}$$

and we have a canonical isomorphism $R \cong R^{\circ\circ}$, again using the α and β constraints of the pseudo adjunctions $N_X, E_X{:}X \dashv X$ of Proposition 3.6. ∎

3.9. Proposition. *For an arrow $R\colon X \to A$ in a cartesian bicategory, with X and A Frobenius, if the $\tilde{d}_R$ and $\tilde{t}_R$ of the units*

$$
\begin{tikzcd}
X \arrow[r, "d_X"] \arrow[d, "R"'] \arrow[dr, phantom, "-\tilde{d}_R\to"] & X \otimes X \arrow[d, "R\otimes R"] \\
A \arrow[r, "d_A"'] & A \otimes A
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "t_X"] \arrow[d, "R"'] \arrow[dr, phantom, "-\tilde{t}_R\to"] & I \arrow[d, "\top"] \\
A \arrow[r, "t_A"'] & I
\end{tikzcd}
$$

are invertible then we can construct squares N_R and E_R

$$
N_R = \begin{tikzcd}
I \arrow[r, "1_I"] \arrow[d, "t_X^*"] \arrow[dd, bend right=60, "N_X"'] \arrow[dr, phantom, "-\tilde{t}_R^*\to"] & I \arrow[d, "t_A^*"'] \arrow[dd, bend left=60, "N_A"] \\
X \arrow[r, "R"] \arrow[d, "d_X"] \arrow[dr, phantom, "-\tilde{d}_R^{-1}\to"] & A \arrow[d, "d_A"'] \\
X \otimes X \arrow[r, "R\otimes R"'] & A \otimes A
\end{tikzcd}
\qquad
E_R = \begin{tikzcd}
X \otimes X \arrow[r, "R\otimes R"] \arrow[d, "d_X^*"] \arrow[dd, bend right=60, "E_X"'] \arrow[dr, phantom, "-\tilde{d}_R^*\to"] & A \otimes A \arrow[d, "d_A^*"'] \arrow[dd, bend left=60, "E_A"] \\
X \arrow[r, "R"] \arrow[d, "t_X"] \arrow[dr, phantom, "-\tilde{t}_R^{-1}\to"] & A \arrow[d, "t_A"'] \\
I \arrow[r, "1_I"'] & I
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "R"] \arrow[d, "X"'] \arrow[dr, phantom, "-R\to"] & A \arrow[d, "A"] \\
X \arrow[r, "R"'] & A
\end{tikzcd}
$$

where $\tilde{t}_R^*$ is the mate of $\tilde{t}_R$ and $\tilde{d}_R^*$ is the mate of $\tilde{d}_R$, which when tensored with the identity square R, above, satisfy the following equations (in which $\otimes$ is suppressed):

$$
R = \begin{tikzcd}
X \arrow[r, "R"] \arrow[d, "N_X X"'] \arrow[dd, bend right=60, "X"'] \arrow[dr, phantom, "\xrightarrow{N_R R}"] & A \arrow[d, "N_A A"] \arrow[dd, bend left=60, "A"] \\
\cong XXX \arrow[r, "RRR"] \arrow[d, "XE_X"'] \arrow[dr, phantom, "\xrightarrow{RE_R}"] & AAA \cong \arrow[d, "AE_A"] \\
X \arrow[r, "R"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "R"] \arrow[d, "XN_X"'] \arrow[dd, bend right=60, "X"'] \arrow[dr, phantom, "\xrightarrow{RN_R}"] & A \arrow[d, "AN_A"] \arrow[dd, bend left=60, "A"] \\
\cong XXX \arrow[r, "RRR"] \arrow[d, "E_X X"'] \arrow[dr, phantom, "\xrightarrow{E_R R}"] & AAA \cong \arrow[d, "E_A A"] \\
X \arrow[r, "R"'] & A
\end{tikzcd}
= R \tag{2}
$$

Proof. The vertical edges of the diagrams have been clarified in Proposition 3.6. For the rest it suffices for each equation to expand N_R and E_R, verify the following equalities

$$
\begin{tikzcd}
& XX \arrow[rr, "RR"] \arrow[dl, "d^*"'] \arrow[dr, "dX"] & \arrow[d, phantom, "-\tilde{d}_R^{-1}R\to"] & AA \arrow[dr, "dA"] & \\
X \arrow[dr, "d"'] \arrow[rr, phantom, "-\delta_0\to"] & & XXX \arrow[rr, "RRR"] \arrow[dl, "Xd^*"] & \arrow[d, phantom, "-R\tilde{d}_R^*\to"] & AAA \arrow[dl, "Ad^*"] \\
& XX \arrow[rr, "RR"'] & & AA &
\end{tikzcd}
=
\begin{tikzcd}
& XX \arrow[rrr, "RR"] \arrow[dl, "d^*"'] & \arrow[d, phantom, "-\tilde{d}_R^*\to"] & & AA \arrow[dl, "d^*"'] \arrow[dr, "dA"] & \\
X \arrow[rrr, "R"] \arrow[dr, "d"'] & & \arrow[d, phantom, "-\tilde{d}_R^{-1}\to"] & A \arrow[dr, "d"] \arrow[rr, phantom, "-\delta_0\to"] & & AAA \arrow[dl, "Ad^*"] \\
& XX \arrow[rrr, "RR"] & & & AA &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & XX \arrow[rr, "RR"] \arrow[dl, "d^*"'] \arrow[dr, "Xd"] & & AA \arrow[dr, "Ad"] & \\
X \arrow[dr, "d"'] \arrow[rr, phantom, "-\delta_1\rightarrow"] & & XXX \arrow[rr, "RRR"] \arrow[dl, "d^*X"] \arrow[rr, phantom, shift left=4, "-R\tilde{d}_R^{-1}\rightarrow"] \arrow[rr, phantom, shift right=4, "-\tilde{d}_R^*R\rightarrow"'] & & AAA \arrow[dl, "d^*A"] \\
 & XX \arrow[rr, "RR"'] & & AA &
\end{tikzcd}
=
\begin{tikzcd}
 & XX \arrow[rr, "RR"] \arrow[dl, "d^*"'] & & AA \arrow[dl, "d^*"'] \arrow[dr, "Ad"] & \\
X \arrow[dr, "d"'] \arrow[rr, "R"] \arrow[rr, phantom, shift left=4, "-\tilde{d}_R^*\rightarrow"] \arrow[rr, phantom, shift right=4, "-\tilde{d}_R^{-1}\rightarrow"'] & & A \arrow[dr, "d"'] \arrow[rr, phantom, "-\delta_1\rightarrow"] & & AAA \arrow[dl, "d^*A"] \\
 & XX \arrow[rr, "RR"] & & AA &
\end{tikzcd}
$$

and use such further equalities as

$$
R =
\begin{tikzcd}
X \arrow[r, "R"] \arrow[d, "d"'] \arrow[dd, bend right=60, "X"', "\cong"] \arrow[dr, phantom, "\xrightarrow{\tilde{d}_R^{-1}}"] & A \arrow[d, "d"] \arrow[dd, bend left=60, "A", "\cong"'] \\
XX \arrow[r, "RR"] \arrow[d, "Xt"'] \arrow[dr, phantom, "\xrightarrow{R\tilde{t}_R^{-1}}"] & AA \arrow[d, "At"] \\
X \arrow[r, "R"'] & A
\end{tikzcd}
\qquad
R =
\begin{tikzcd}
X \arrow[r, "R"] \arrow[d, "t^*X"'] \arrow[dd, bend right=60, "X"', "\cong"] \arrow[dr, phantom, "\xrightarrow{\tilde{t}_R^*R}"] & A \arrow[d, "t^*A"] \arrow[dd, bend left=60, "A", "\cong"'] \\
XX \arrow[r, "RR"] \arrow[d, "d^*"'] \arrow[dr, phantom, "\xrightarrow{\tilde{d}_R^*}"] & AA \arrow[d, "d^*"] \\
X \arrow[r, "R"'] & A
\end{tikzcd}
$$

∎

3.10. Every object X of a bicategory with finite products is, essentially uniquely, a pseudo comonoid via d_X and t_X. It follows that every object X in a cartesian bicategory $\mathbf{B}$ is a (pseudo) comonoid (via d_X and t_X) since $\mathbf{M}$ has finite products and the inclusion functor $i{:}\mathbf{M}\longrightarrow\mathbf{B}$ is strongly monoidal. (It is the identity on objects and we observe from Proposition 3.24 of [CKWW] that $f \times g \xrightarrow{\simeq} f \otimes g$ in $\mathbf{B}$.) Similarly, for $R{:}X\longrightarrow A$ in $\mathbf{B}$, R has an essentially unique comonoid structure in $\mathbf{G}$, via $(d_X, \tilde{d}_R, d_A)$ and $(t_X, \tilde{t}_R, t_A)$, since $\mathbf{G}$ has finite products. In fact, given d_X and d_A, $\tilde{d}_R$ is uniquely determined and given t_X and t_A, $\tilde{t}_R$ is uniquely determined. This fact can be reinterpretted to say that $R{:}X\longrightarrow A$ has an essentially unique lax comonoid homomorphism structure via $d_R = (d_X, \tilde{d}_R, d_A)$ and $t_R = (t_X, \tilde{t}_R, t_A)$ which is then a *comonoid homomorphism* if and only if the 2-cells $\tilde{d}_R$ and $\tilde{t}_R$ are invertible. Thus being a comonoid homomorphism is a *property* of an arrow in a cartesian bicategory.

3.11. THEOREM. *For an arrow $R{:}X\longrightarrow A$ in a cartesian bicategory, with X and A Frobenius, the following are equivalent:*

(1) R is a map;

(2) R is a comonoid homomorphism;

(3) $R \dashv R^\circ$.

Proof. (1) implies (2) follows from the fact that d and t are pseudonatural on maps and (3) implies (1) is trivial. So, assuming (2), that R is a comonoid homomorphism, construct N_R and E_R as in Proposition 3.9 and define (suppressing $\otimes$ as usual)

$$
\eta_R = \begin{tikzcd}
 & & A \arrow[dd, "N_XA"] \\
 & X \arrow[ur, "R"] \arrow[dd, "N_XX"'] \arrow[dddd, bend right=60, "X"'] \arrow[r, phantom, "\xrightarrow{N_XR}"] & \\
 & & XXA \arrow[dd, "XRA"] \\
 \arrow[r, phantom, "\cong"] & XXX \arrow[ur, "XXR"] \arrow[dr, "XRR"'] \arrow[dd, "XE_X"'] \arrow[r, phantom, "\cong"] & \\
 & & XAA \arrow[dd, "XE_A"] \\
 & X \arrow[dr, "X"'] \arrow[r, phantom, "\xrightarrow{XE_R}"] & \\
 & & X
\end{tikzcd}
\qquad
\epsilon_R = \begin{tikzcd}
A \arrow[dr, "A"] \arrow[dd, "N_XA"'] & & \\
 & A \arrow[l, phantom, "\xrightarrow{N_RA}"] \arrow[dd, "N_AA"] \arrow[dddd, bend left=60, "A"] & \\
XXA \arrow[dr, "RRA"] \arrow[dd, "XRA"'] & & \\
\arrow[r, phantom, "\cong"] & AAA \arrow[dd, "AE_A"] \arrow[r, phantom, "\cong"] & {} \\
XAA \arrow[ur, "RAA"] \arrow[dd, "XE_A"'] & & \\
 & A \arrow[l, phantom, "\xrightarrow{RE_A}"] & \\
X \arrow[ur, "R"'] & &
\end{tikzcd}
$$

where we note that both three-fold vertical composites are the arrow R°, $N_XR = 1_{N_X} \otimes 1_R$ and $RE_A = 1_R \otimes 1_{E_A}$ are isomorphisms while $XE_R = 1_{1_X} \otimes E_R$ and $N_RA = N_R \otimes 1_{1_A}$. When η_R and ϵ_R are pasted at R° the result is

$$
\begin{tikzcd}
 & A \arrow[dr, "A"] \arrow[dd, "N_XA"] & \\
X \arrow[ur, "R"] \arrow[dd, "N_XX"'] \arrow[dddd, bend right=60, "X"'] \arrow[r, phantom, "\xrightarrow{N_XR}"] & & A \arrow[l, phantom, "\xrightarrow{N_RA}"] \arrow[dd, "N_AA"] \arrow[dddd, bend left=60, "A"] \\
 & XXA \arrow[dr, "RRA"] & \\
XXX \arrow[ur, "XXR"] \arrow[dr, "XRR"'] \arrow[dd, "XE_X"'] \arrow[r, phantom, "\cong"] & \cong & AAA \arrow[dd, "AE_A"] \arrow[l, phantom, "\cong"] \\
 & XAA \arrow[ur, "RAA"] \arrow[dd, "XE_A"] & \\
X \arrow[dr, "X"'] \arrow[r, phantom, "\xrightarrow{XE_R}"] & & A \arrow[l, phantom, "\xrightarrow{RE_A}"] \\
 & X \arrow[ur, "R"'] &
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "R"] \arrow[d, "N_XX"'] \arrow[dd, bend right=60, "X"'] & A \arrow[d, "N_AA"] \arrow[dd, bend left=60, "A"] \\
XXX \arrow[r, "RRR"] \arrow[d, "XE_X"'] \arrow[u, phantom, "\cong"] & AAA \arrow[d, "AE_A"] \arrow[u, phantom, "\cong"] \\
X \arrow[r, "R"'] & A
\end{tikzcd}
= R
$$

$$
N_RR \text{ (upper square)}, \quad RE_R \text{ (lower square)}
$$

the first equality from functoriality of $\otimes$, the second equality being the first equation of (2) of Proposition 3.9. To complete the proof that we have an adjunction $\eta_R, \epsilon_R: R \dashv R^\circ$ we must show that when η_R is pasted to ϵ_R at R the result is R°. To aid readability we

draw as commutative as many regions as possible. Consider:

$$
\begin{tikzcd}
& A \arrow[rr, "N_A A"] \arrow[rrrr, bend left, "A"] \arrow[d, phantom, "\Uparrow N_R A"] & & AAA \arrow[rr, "AE_A"] & & A \arrow[rr, "N_X A"] \arrow[rrrrrr, bend left, "R^\circ"] & & XXA \arrow[rr, "XRA"] & & XAA \arrow[rr, "XE_A"] & & X \\
A \arrow[ur, "A"] \arrow[rr, "N_X A"'] \arrow[rrrrrr, bend right, "R^\circ"'] & & XXA \arrow[ur, "RRA"] \arrow[rr, "XRA"'] & & XAA \arrow[ul, "RAA"'] \arrow[rr, "XE_A"'] & & X \arrow[ul, "R"'] \arrow[rr, "N_X X"'] \arrow[rrrr, bend right, "X"'] & & XXX \arrow[ul, "XXR"'] \arrow[ur, "XRR"] \arrow[rr, "XE_X"'] & & X \arrow[ur, "X"'] \arrow[u, phantom, "\Uparrow XE_R"] &
\end{tikzcd}
$$

(which is the requisite pasting rotated 90 degrees counterclockwise). Rearrange it as below:

$$
\begin{tikzcd}
A \arrow[r, "A"] \arrow[d, "N_X A"'] & A \arrow[r, "N_X A"] \arrow[d, "N_A A"] & XXA \arrow[r, "XRA"] \arrow[d, "XXN_A A"] & XAA \arrow[d, "XAN_A A"] \arrow[ddrr, "XAA"] & & \\
XXA \arrow[r, "RRA"'] \arrow[ur, phantom, "\xrightarrow{N_R A}"] \arrow[drr, "N_X XXA"] \arrow[ddrr, "XXA"'] & AAA \arrow[r, "N_X AAA"'] & XXAAA \arrow[r, "XRAAA"'] & XAAAA \arrow[drr, "XAAE_A"] & & \\
& & XXXXA \arrow[r, "XXXRA"] \arrow[u, "XXRRA"] \arrow[ur, "XRRRA"] \arrow[d, "XE_X XA"] & XXXAA \arrow[r, "XXXE_A"] \arrow[u, "XRRAA"] \arrow[d, "XE_X AA"] & XXX \arrow[r, "XRR"] \arrow[d, "XE_X"'] \arrow[dr, phantom, "\xrightarrow{XE_R}"] & XAA \arrow[d, "XE_A"] \\
& & XXA \arrow[r, "XRA"'] & XAA \arrow[r, "XE_A"'] & X \arrow[r, "X"'] & X
\end{tikzcd}
$$

(3)

The following prism commutes:

$$
\begin{tikzcd}
A \arrow[r, "A"] \arrow[d, "N_X A"'] \arrow[drr, "N_X A"] \arrow[dr, phantom, "\xrightarrow{N_R A}"] & A \arrow[d, "N_A A"] \arrow[drr, "N_X A"] & & & \\
XXA \arrow[r, "RRA"'] \arrow[drr, "N_X XXA"'] & AAA \arrow[drr, "N_X AAA"] & XXA \arrow[r, "XXA"] \arrow[d, "XXN_X A"] \arrow[drr, "XRA"] \arrow[dr, phantom, "\xrightarrow{XXN_R A}"] & XXA \arrow[d, "XXN_A A"] \arrow[dr, "XRA"] & \\
& & XXXXA \arrow[r, "XXRRA"'] \arrow[drr, "XRRRA"'] & XXAAA \arrow[dr, "XRAAA"] \arrow[r, phantom, "\xrightarrow{XRN_R A}"] & XAA \arrow[d, "XAN_A A"] \\
& & & & XAAAA
\end{tikzcd}
$$

Replace the top three squares of (3) above by the two front faces of the prism. Employ a similar commuting prism to replace the bottom three squares of (3) and obtain:

$$
\begin{tikzcd}
A \arrow[r, "N_X A"] \arrow[d, "N_X A"'] & XXA \arrow[r, "XRA"] \arrow[d, "XXN_X A"'] \arrow[dr, phantom, "\xrightarrow{XRN_R A}"] & XAA \arrow[d, "XAN_A A"'] \arrow[dr, "XAA"] & \\
XXA \arrow[r, "N_X XXA"] \arrow[dr, "XXA"'] & XXXXA \arrow[r, "XRRRA"] \arrow[d, "XE_X XA"'] \arrow[dr, phantom, "\xrightarrow{XE_R RA}"] & XAAAA \arrow[r, "XAAE_A"] \arrow[d, "XE_A AA"] & XAA \arrow[d, "XE_A"] \\
& XXA \arrow[r, "XRA"'] & XAA \arrow[r, "XE_A"'] & X
\end{tikzcd}
$$

$$
= \quad
\begin{tikzcd}
A \arrow[r, "N_X A"] \arrow[d, "A"'] & XXA \arrow[r, "XRA"] \arrow[d, "XXA"'] & XAA \arrow[r, "XE_A"] \arrow[d, "XAA"] & X \arrow[d, "X"] \\
A \arrow[r, "N_X A"'] & XXA \arrow[r, "XRA"'] & XAA \arrow[r, "XE_A"'] & X
\end{tikzcd}
\quad = R^\circ
$$

where the penultimate equality is obtained from the second equation of (2) of Proposition 3.9 by tensoring it on the left by X and on the right by A and applying the result to the two middle squares of the penultimate pasting. ∎

3.12. From Theorem 3.11 it follows that for a map $f{:}X{\to}A$, with X and A Frobenius in a cartesian bicategory, we have $f^* \cong f^\circ$ and we may as well write $f^* = f^\circ$ for our specified right adjoints in this event and use the explicit formula for f° when it is convenient to do so.

3.13. Theorem. *If A is a Frobenius object in a cartesian bicategory $\mathbf{B}$, then, for all T in $\mathbf{B}$, the hom-category $\mathbf{M}(T, A)$ is a groupoid.*

We will break the proof of Theorem 3.13 into a sequence of lemmas and employ the notation of 2.3.

3.14. Lemma. *With reference to the 2-cell δ_1 in Definition 3.1,*

$$dd^* \xrightarrow{\cong} (p^* \wedge r^*)(p \wedge r) \quad \text{and} \quad (d^* \otimes X)(X \otimes d) \xrightarrow{\cong} p^*p \wedge p^*r \wedge r^*r$$

and these canonical isomorphisms identify δ_1 with $(\pi\pi, \pi\rho, \rho\rho)$. Here the components are horizontal composites of the local product projection 2-cells. For example, $\pi\rho$ is

$$\begin{tikzcd} A \otimes A \arrow[r, bend left, "p\wedge r"] \arrow[r, bend right, "r"'] \arrow[r, phantom, "\Downarrow\rho"] & A \arrow[r, bend left, "p^*\wedge r^*"] \arrow[r, bend right, "p^*"'] \arrow[r, phantom, "\Downarrow\pi"] & A \otimes A \end{tikzcd}$$

We will write

$$\delta = (\pi\pi, \pi\rho, \rho\rho) \colon (p^* \wedge r^*)(p \wedge r) \to p^*p \wedge p^*r \wedge r^*r \colon A \otimes A \to A \otimes A \tag{4}$$

Proof. We have

$$p \wedge r \cong d^*(p \otimes r)d \cong d^*(p, r) \cong d^* 1_{A\otimes A} = d^*$$

and

$$p^* \wedge r^* \cong d^*(p^* \otimes r^*)d \cong d^*(p \otimes r)^* d \cong (p, r)^* d \cong 1^*_{A\otimes A} d = d$$

so that $dd^* \cong (p^* \wedge r^*)(p \wedge r)$. To exhibit the other isomorphism of the statement we will write $d_3 : A \otimes A \to (A \otimes A) \otimes (A \otimes A) \otimes (A \otimes A)$ for the three-fold diagonal map $(1_{A\otimes A}, 1_{A\otimes A}, 1_{A\otimes A})$ and then

$$p^*p \wedge p^*r \wedge r^*r \cong d_3^*(p^*p \otimes p^*r \otimes r^*r)d_3 \cong d_3^*(p^* \otimes p^* \otimes r^*)(p \otimes r \otimes r)d_3 \cong (d^* \otimes A)(A \otimes d)$$

■

Of course $\delta = (\pi\pi, \pi\rho, \rho\rho)$ in (4) of the Lemma is invertible if and only if A is Frobenius. We will write

$$\nu = \rho\pi \colon (p^* \wedge r^*)(p \wedge r) \to r^*p \colon A \otimes A \to A \otimes A$$

for the "other" horizontal composite of projections and for A Frobenius we define μ as the unique 2-cell $(\nu.\delta^{-1})$ making commutative

$$\begin{tikzcd} (p^* \wedge r^*)(p \wedge r) \arrow[rr, "\delta"] \arrow[dr, "\nu"'] & & p^*p \wedge p^*r \wedge r^*r \arrow[dl, "\mu"] \\ & r^*p & \end{tikzcd} \tag{5}$$

We remark that a local product of maps is not generally a map. (In the case of the bicategory of relations a local product of maps is a partial map.) Observe though that if A is such that the maps $d: A \to A \otimes A$ and $t: A \to I$ have right adjoints *in* $\mathbf{M}$ then A is a cartesian object in $\mathbf{M}$ in the terminology of [CKW] and [CKVW]. In this case $p \wedge r{:}A \otimes A \to A$ is the map that provides "internal" binary products for A.

For maps $f, g{:}T \rightrightarrows A$ we write, as in 2.3, $A(f,g)$ for the composite f^*g and observe that the following three kinds of 2-cells are in natural bijective correspondence

$$\begin{tikzcd} T \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & A \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, bend left, "1_T"] \arrow[r, bend right, "A(f,g)"'] \arrow[r, phantom, "\Downarrow\widehat{\alpha}"] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, bend left, "g^*"] \arrow[r, bend right, "f^*"'] \arrow[r, phantom, "\Downarrow\alpha^*"] & A \end{tikzcd}$$

We have

3.15. Lemma. *The hom-category $\mathbf{M}(T,A)$ can be equivalently described as the category whose objects are the maps $f{:}T \to A$ and whose hom-sets $\mathbf{M}(T,A)(f,g)$ are the sets $\mathbf{M}(T,T)(1_T, A(f,g))$ with composition given by pasting composites of the form*

$$\begin{tikzcd} T \arrow[r, bend left, "1_T"] \arrow[r, bend right, "A(g,h)"'] \arrow[r, phantom, "\Downarrow\widehat{\beta}"] \arrow[rr, bend right=80, "A(f,h)"'] & T \arrow[r, bend left, "1_T"] \arrow[r, bend right, "A(f,g)"'] \arrow[r, phantom, "\Downarrow\widehat{\alpha}"] \arrow[d, phantom, "\Downarrow f^*\epsilon_g h"] & T \\ & {} & \end{tikzcd}$$

Proof. It is a simple exercise with mates to show that the pasting composite displayed is $\widehat{\beta\alpha}$. We note that $\widehat{1_f} = \eta_f$. ∎

3.16. Lemma. *For objects f, h, g, k of $\mathbf{M}(T,A)$, the whisker composite*

$$\begin{tikzcd} T \arrow[r, "{(g,k)}"] & A \otimes A \arrow[r, bend left, "(p^*\wedge r^*)(p\wedge r)"] \arrow[r, bend right, "p^*p\wedge p^*r\wedge r^*r"'] \arrow[r, phantom, "\Downarrow \delta=(\pi\pi,\pi\rho,\rho\rho)"] & A \otimes A \arrow[r, "{(f,h)^*}"] & T \end{tikzcd}$$

being in the notation of 2.3

$$(p^* \wedge r^*)(p \wedge r)((f,h)(g,k)) \xrightarrow{\delta((f,h)(g,k))} (p^*p \wedge p^*r \wedge r^*r)((f,h)(g,k))$$

is

$$
\begin{tikzcd}
T \arrow[r, "g\wedge k"] \arrow[rr, bend right, "A(f,g)\wedge A(f,k)\wedge A(h,k)"'] & A \arrow[r, "f^*\wedge h^*"] \arrow[d, dashed, "(\pi\pi,\pi\rho,\rho\rho)"] & T \\
 & {} &
\end{tikzcd}
$$

In fact, $(p \wedge r)(g,k) \cong g \wedge k$ *and* $(f,h)^*(p^* \wedge r^*) \cong f^* \wedge h^*$*.*

Proof. We have

$$(p \wedge r)(g,k) \cong d^*(p \otimes r)d(g,k) \cong d^*(p \otimes r)((g,k) \otimes (g,k))d \cong d^*(g \otimes k)d \cong g \wedge k$$

while

$$(f,h)^*(p^* \wedge r^*) \cong (f,h)^* d^*(p^* \otimes r^*)d \cong ((p \otimes r)d(f,h))^* d \cong ((f \otimes h)d)^* d \cong f^* \wedge h^*$$

On the other hand, precomposing with maps and postcomposing with pams preserves local products so that we have

$$
\begin{aligned}
(f,h)^*(p^*p \wedge p^*r \wedge r^*r)(g,k) &\cong (f,h)^*(p^*p)(g,k) \wedge (f,h)^*(p^*r)(g,k) \wedge (f,h)^*(r^*r)(g,k) \\
&\cong f^*g \wedge f^*k \wedge h^*k
\end{aligned}
$$

Assembling these results in hom-notation gives the statement. ∎

The whisker composite in Lemma 3.16 should be thought of as the *instantiation* of δ at $((f,h)(g,k))$ and we have been deliberately selective in mixing our notations in the concluding diagram of the statement; $(\pi\pi, \pi\rho, \rho\rho)$ being more informative than $\delta((f,h)(g,k))$. If we instantiate the rest of diagram (5) at $((f,h)(g,k))$, which is to say whisker with $(f,h)^*(-)(g,k)$, then the result is clearly the lower triangle below.

$$
\begin{tikzcd}
 & 1_T \arrow[dl, "\Xi"'] \arrow[dr, "(\alpha,\beta,\gamma)"] & \\
(f^* \wedge h^*)(g \wedge k) \arrow[rr, "(\pi\pi,\pi\rho,\rho\rho)"] \arrow[dr, "\rho\pi"'] & & A(f,g) \wedge A(f,k) \wedge A(h,k) \arrow[dl, "\mu((f,h),(g,k)"] \\
 & A(h,g) &
\end{tikzcd}
\tag{6}
$$

In the top triangle above it is clear that a 1_T-element of $A(f,g) \wedge A(f,k) \wedge A(h,k)$ is exactly an “S” shaped configuration in $\mathbf{M}(T,A)$ of the form

$$
\begin{tikzcd}
f \arrow[r, "\alpha"] \arrow[dr, "\beta"] & g \\
h \arrow[r, "\gamma"'] & k
\end{tikzcd}
$$

For A Frobenius we will be interested in lifting 1_T-elements of $A(f,g)\wedge A(f,k)\wedge A(h,k)$ though the isomorphism

$$(\pi\pi,\pi\rho,\rho\rho){:}(f^*\wedge h^*)(g\wedge k)\to A(f,g)\wedge A(f,k)\wedge A(h,k)$$

As we discussed in 2.3, we do not have precise knowledge of general 1_T-elements Ξ of

$$(f^*\wedge h^*)(g\wedge k)=((p^*\wedge r^*)(p\wedge r))((f,h)(g,k))$$

but those obtained by pasting a 1_T-element of $(p^*\wedge r^*)((f,h),x)$ to a 1_T-element of $(p\wedge r)(x,(g,k))$, for some $x{:}T\to A$ present no difficulty. (Here, $p^*\wedge r^*$ is the S and $p\wedge r$ is the R of 2.3.) Since

$$(p^*\wedge r^*)((f,h),x)=(f,h)^*(p^*\wedge r^*)x\cong(f^*\wedge h^*)x\cong f^*x\wedge h^*x=A(f,x)\wedge A(h,x)$$

and

$$(p\wedge r)(x,(g,k))=x^*(p\wedge r)(g,k)\cong x^*(g\wedge k)\cong x^*g\wedge x^*k=A(x,g)\wedge A(x,k)$$

(where we have used Lemma 3.16 in each derivation) we see that these special 1_T-elements of $(f^*\wedge h^*)(g\wedge k)$ are given by (equivalence classes of) "X" shaped configurations in $\mathbf{M}(T,A)$ of the form

$$\begin{tikzcd} f \arrow[dr, "\xi"] & & g \\ & x \arrow[ur, "\eta"'] \arrow[dr, "\omega"'] & \\ h \arrow[ur, "\zeta"] & & k \end{tikzcd}$$

It is convenient to write such a 1_T-element of $(f^*\wedge h^*)(g\wedge k)$ as the following pasting composite

$$\begin{tikzcd} & T \arrow[rr, "1_T"] \arrow[dl, "1_T"'] \arrow[dr, "x"] \arrow[d, phantom, "\Downarrow(\eta,\omega)"] & \arrow[d, phantom, "\Downarrow\eta_x"] & T \arrow[dr, "1_T"] \arrow[d, phantom, "\Downarrow(\xi^*,\zeta^*)"] & \\ T \arrow[rr, "g\wedge k"'] & {} & A \arrow[ur, "x^*"] \arrow[rr, "f^*\wedge h^*"'] & {} & T \end{tikzcd} \tag{7}$$

Invertibility of $\delta=(\pi\pi,\pi\rho,\rho\rho){:}(f^*\wedge h^*)(g\wedge k)\to X(f,g)\wedge X(f,k)\wedge X(h,k)$ tells us that, for every "S" configuration (α,β,γ), there is a unique 1_T-element Ξ of $(f^*\wedge h^*)(g\wedge k)$ such that $\delta\Xi=(\alpha,\beta,\gamma)$. When, as in several classical situations, every 1_T-element Ξ comes from an "X" configuration we have motivation for the colloquial name "S"="X" for the Frobenius condition. (In fact one says "S"="X"="Z" when the second "equation" is not derivable from the first but we have Lemma 3.2.)

3.17. Lemma. *For a 1_T-element Ξ (see (6)) arising from an "X" configuration as in (7), $\delta\Xi=(\eta\xi,\omega\xi,\omega\zeta)$ and $\nu\Xi=\eta\zeta$.*

Proof. For $\delta\Xi$ we treat the components separately. For the first, we paste

$$
\begin{tikzcd}
T \arrow[r, bend left=50, "g\wedge k"] \arrow[r, bend right=50, "g"'] \arrow[r, phantom, "\Downarrow\pi"] & A \arrow[r, bend left=50, "f^*\wedge h^*"] \arrow[r, bend right=50, "f^*"'] \arrow[r, phantom, "\pi\Downarrow"] & T
\end{tikzcd}
$$

to (7) and obtain the 1_T-element

$$
\begin{tikzcd}
& T \arrow[rr, "1_T"] \arrow[dl, "1_T"'] \arrow[dr, "x"] \arrow[d, phantom, "\Downarrow\eta"] & \arrow[d, phantom, "\Downarrow\eta_x"] & T \arrow[dr, "1_T"] \arrow[d, phantom, "\Downarrow\xi^*"] & \\
T \arrow[rr, "g"'] & & A \arrow[ur, "x^*"] \arrow[rr, "f^*"'] & & T
\end{tikzcd}
$$

of $A(f,g)$. To see this as a 2-cell $f \to g$ paste onto it

$$
\begin{tikzcd}
A \arrow[r, "f^*"'] \arrow[rr, bend right=50, "1_A"'] & T \arrow[r, "f"] \arrow[d, phantom, "\Downarrow\epsilon_f"] & A \\
& {} &
\end{tikzcd}
$$

(at f^*) which is the "unhatting" bijection and observe that the result is $\eta\xi : f \to g$. For the second, first paste (π, ρ) and then paste ϵ_f. For the third, first paste (ρ, ρ) and then paste ϵ_h. For $\nu\xi$, paste (ρ, π) to (7) and then paste ϵ_h (at h^*). ∎

The 2-cell μ of (5) when instantiated as in (6) provides a completion of "S" configurations, as by the dotted arrow below. (It ultimately has the air of a Malcev operation.)

$$
\begin{tikzcd}
f \arrow[r] \arrow[dr] & g \\
h \arrow[r] \arrow[ur, dotted] & k
\end{tikzcd}
$$

In particular, given a 2-cell $\alpha : f \to g$ we have the "S" configuration $(1_f, \alpha, 1_g)$ and we write $\alpha^\dagger = \mu(1, \alpha, 1)$.

3.18. Lemma. $\alpha^\dagger = \alpha^{-1}$

Proof. The composite $\alpha\alpha^\dagger$ is the clockwise composite $1_T \to A(g,g)$ in the following commutative diagram.

$$\begin{tikzcd}
 & A(f,f) \wedge A(f,g) \wedge A(g,g) \arrow[r, "\mu"] \arrow[dd, "A(f,\alpha)\wedge A(f,g)\wedge A(g,g)"] & A(g,f) \arrow[dd, "A(g,\alpha)"] \\
1_T \arrow[ur, "(1,\alpha,1)"] \arrow[dr, "(\alpha,\alpha,1)"'] & & \\
 & A(f,g) \wedge A(f,g) \wedge A(g,g) \arrow[r, "\mu"'] & A(g,g)
\end{tikzcd}$$

We show that $\alpha\alpha^\dagger = 1_g$ by evaluating the counterclockwise composite. While we do not know if an "X" configuration gives rise to the 1_T-element $\delta^{-1}(1,\alpha,1)$ we *do* know that $(\alpha,\alpha,1)$ arises from the "X" configuration

$$\begin{tikzcd}
f \arrow[dr, "\alpha"] & & g \\
 & g \arrow[ur, "1_g"'] \arrow[dr, "1_g"] & \\
g \arrow[ur, "1_g"] & & g
\end{tikzcd}$$

because, writing Ξ for the 1_T-element arising as in (7) we have, by Lemma 3.17, $\delta\Xi = (\alpha,\alpha,1)$. It follows using (6) and again Lemma 3.17 that

$$\alpha\alpha^\dagger = \mu(\alpha,\alpha,1) = \nu\Xi = 1_g$$

Similarly, the composite $\alpha^\dagger\alpha$ is the clockwise composite in the commutative diagram.

$$\begin{tikzcd}
 & A(f,f) \wedge A(f,g) \wedge A(g,g) \arrow[r, "\mu"] \arrow[dd, "A(f,f)\wedge A(f,g)\wedge A(\alpha,g)"] & A(g,f) \arrow[dd, "A(\alpha,f)"] \\
1_T \arrow[ur, "(1,\alpha,1)"] \arrow[dr, "(1,\alpha,\alpha)"'] & & \\
 & A(f,f) \wedge A(f,g) \wedge A(f,g) \arrow[r, "\mu"'] & A(f,f)
\end{tikzcd}$$

The rest of the proof proceeds as above after observing that $(1, \alpha, \alpha)$ arises from the "X" configuration

$$
\begin{tikzcd}
f \arrow[dr, "1_f"] & & f \\
& f \arrow[ur, "1_f"'] \arrow[dr, "\alpha"'] & \\
f \arrow[ur, "1_f"] & & g
\end{tikzcd}
$$

■

This completes the proof of Theorem 3.13.

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